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The Existence of Phase V in the Mandelbrot Percolation Process

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Dekking and Meester defined six phases for a subclass of random Cantor sets consisting of those generated by Bernoulli random substitutions. They proved that the random Sierpinski carpet passed through all these phases as p tended from 0 to 1, but they were not able to prove the existence of phase V in the Mandelbrot percolation process. In this paper, we accomplish the proof by improving their methods.

KEY WORDS: Mandelbrot percolation; random substitution.

1. ORIGIN OF THE PROBLEM

1.1. Random Substitutions and Random Cantor Sets

Definition 1.1. We call σ: $\{0, 1\} \to \{0, 1\}^{N\times N}$ a random substitution if:

(i) We have

$$\sigma(0) = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

(ii) $\sigma(1) \in \{U_1,..., U_r\}$, $U_1,..., U_r$ are $N\times N$ 0–1-valued matrices, and there are positive numbers $p_1,..., p_r$ with $p_1 + \cdots + p_r = 1$ such that $\mathbb{P}[\sigma(1) = U_i] = p_i$, for $1 \leqslant i \leqslant r$.

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Iteration is defined as follows: $\sigma^{n+1}(1)$ is obtained by replacing all 0's in $\sigma^n(1)$ by $\sigma(0)=0$, and all 1's by independent random matrices distributed as $\sigma(1)$.

Iterating σ n times leads to an $N^n \times N^n$ random matrix $W^n = \sigma^n(1)$. Let $I^n_{kl} = I^n_k \times I^n_l$, where $I^n_k = [(k-1)N^{-n}, kN^{-n}]$, for $1 \leqslant k, l \leqslant N^n$. We call such a set I^n_{kl} a *level-n square*. We now consider the sets

$$A_n = \bigcup_{k,l} \{I^n_{kl} : W^n_{kl} = 1\}$$

The A_n decrease to a compact subset A of the unit square, which is called a random Cantor set.

1.2. Bernoulli Substitution and Mandelbrot Percolation Process

Definition 1.2. We call a random substitution σ a *Bernoulli* (random) substitution with parameter p if there is a set J of indices (k, l) such that $\sigma(1)_{kl}=0$ if $(k, l) \notin J$ and $\mathbb{P}[\sigma(1)_{kl}=1] = 1 - \mathbb{P}[\sigma(1)_{kl}=0] = p$ for $(k, l) \in J$, independent of all other entries of $\sigma(1)$.

Example 1.3. Given $N=3$, $J=\{(k, l) : k \neq 2 \text{ or } l \neq 2\}$, the limiting set A is called a *random Sierpinski carpet*.

Example 1.4. Given $N>1$ and J includes the full set of indices, A is called a *Mandelbrot percolation process*.

For a random Cantor set A, A percolates means that A contains a connected component which has a nonempty intersection with the left and right side, of the unit square.

1.3. Morphology of Random Cantor Sets

Let A be a random Cantor set in the unit square generated by a Bernoulli substitution. πA denotes the projection of A onto the x axis. λ denotes the Lebesgue measure, and $\dim(\cdot)$ denotes the Hausdorff dimension of a set. The six states of A are defined as follows:

I. $A = \varnothing$, almost surely.

II. $\mathbb{P}[A \neq \varnothing] > 0$, but $\dim(\pi A) = \dim(A)$.

III. $\dim(\pi A) < \dim(A)$ a.s. given $A \neq \varnothing$, but $\lambda(\pi A) = 0$, a.s.

IV. $0 < \lambda(\pi A) < 1$ a.s. given $A \neq \varnothing$.

V. $\mathbb{P}[\lambda(\pi A) = 1] > 0$, but A does not percolate a.s.

VI. A percolates with positive probability.

The following result tells us when a random Cantor set is in one of the first three phases. Let m_l be the average number of ones in the lth column of $\sigma(1)$, i.e.,

$$m_l = \sum_{i=1}^{r} p_i \sum_{k=1}^{N} U_i(k, l)$$

Theorem 1.5. Let A be a random Cantor set. Then:

1. $A = \varnothing$, a.s., iff $\sum_{l=1}^{N} m_l \leqslant 1$ [unless $\sigma(1)$ contains exactly one 1 a.s.].
2. $\dim(\pi A) = \dim(A)$ iff $\sum_{l=1}^{N} m_l \log m_l \leqslant 0$.
3. $\lambda(\pi A) = 0$ iff $\sum_{l=1}^{N} \log m_l \leqslant 0$.

Proof. See Dekking and Meester,[1] Theorem 2.1. ∎

In the following, we will only consider the Mandelbrot percolation process, i.e., we will always assume that $N = 3$, J includes all the indices.

Example 1.6. For Mandelbrot percolation, we have $m_1 = m_2 = m_3 = 3p$. Hence $A = \varnothing$ a.s. for $0 \leqslant p \leqslant 1/9$, A is in phase II for $1/9 < p \leqslant 1/3$, there is no phase III: at $p = 1/3$, A passes from II to IV.

1.4. Multivalued and Random Substitutions

Let $\Phi(1)$ be a nonempty set of words of length N, such that $0 \cdots 0 \notin \Phi(1)$. Let $\Phi(0)$ be the complement of $\Phi(1)$ in $\{0, 1\}^N$. For words w and w', ww' denotes the concatenation of w and w'. Furthermore, we define the following sets:

$$\Phi(vw) = \{v'w' \mid v' \in \Phi(v),\ w' \in \Phi(w)\}$$

$$\Phi^n(1) = \{\Phi(w) \mid w \in \Phi^{n-1}(1)\}, \qquad n \geqslant 2$$

The set $\Phi^n(0)$ is defined analogously.

Lemma 1.7. $\Phi^n(0) = \{0, 1\}^{N^n} \backslash \Phi^n(1)$, for all $n \geqslant 1$.

Proof. See Dekking and Meester,[1] Lemma 3.1. ∎

Proposition 1.8. Let σ_p be a Bernoulli random substitution and define

$$\pi_0(p) = 1$$

$$\pi_n(p) = \mathbb{P}[\sigma_p^n(1) \in \Phi^n(1)], \qquad n \geqslant 1$$

Then we have, for all $n \geqslant 0$,

$$\pi_{n+1}(p) = \pi_1(p\pi_n(p))$$

Proof. See Dekking and Meester,[1] Proposition 3.3. ∎

The set $\Phi(1)$ is said to be *increasing* if the following is true: if $w = w_1 \cdots w_N \in \Phi(1)$ and $w_i = 0$ for some i, then $w' = w_1 \cdots w_{i-1} 1 w_{i+1} \cdots w_N \in \Phi(1)$. If $\Phi(1)$ is increasing, it follows from Grimmett,[2] Section 2.5, that $\pi_1(p)$ is an increasing function in p. We define an iteration function G_p as follows:

$$G_p(x) = \pi_1(px), \qquad x, p \in [0, 1]$$

It follows from Proposition 1.8 that $\pi_{n+1}(p) = G_p(\pi_n(p))$ and we obtain $\pi_n(p) = G_p^n(1)$, $n \geqslant 1$. If $\Phi(1)$ is increasing, $G_p(\cdot)$ is increasing and hence $\pi^{\Phi}(p) = \lim_{n \to \infty} \pi_n(p)$ is equal to the largest fixed point of G_p. We define

$$p_c(\Phi) = \inf\{p \mid \pi^{\Phi}(p) > 0\}$$

Lemma 1.9. Consider a Bernoulli random substitution and let $\Phi(1)$ be an increasing set. Suppose that $(\partial/\partial p)\, \pi_1(p)|_{p=1} < 1$. Then $p_c(\Phi) < 1$ and $p_c(\Phi)$ is equal to the smallest p for which the following system has a solution:

$$\begin{cases} G_p(x) = x \\ \dfrac{\partial}{\partial x} G_p(x) = 1 \end{cases} \qquad (*)$$

Proof. See Dekking and Meester,[1] Lemma 3.4. ∎

Lemma 1.10. Let

$$p_{\mathrm{IV,V}} = \inf\{p \colon \mathbb{P}[\lambda(\pi A) = 1] > 0\}$$

Then $p_{\mathrm{IV,V}} \leqslant 0.7307$.

Proof. Take

$$\Phi(1) = \left\{(u_{kl}) \colon 1 \leqslant k, l \leqslant 3, \text{ and } \sum_{k=1}^{3} u_{kl} > 0, 1 \leqslant l \leqslant 3\right\}$$

Then we have $G_p(x) = [1 - (1 - px)^3]^3$.

Using (*) of Lemma 1.9, it follows that

$$x \approx 0.791701$$
$$p = p_c(\Phi) \approx 0.730661 \quad \blacksquare$$

2. IMPROVING THE ESTIMATION OF THE UPPER BOUND OF $p_{\mathrm{IV,V}}$

We call a 0–1-valued word $u = (u_{ij})$, $1 \leqslant i \leqslant N$, $1 \leqslant j \leqslant M$, a fully projected matrix if $\sum_{i=1}^{N} u_{ij} > 0$ for all j. In Lemma 1.10, $\Phi(1)$ corresponds the set of 3×3 fully projected matrices. The defect of this method is that $\Phi^m(1)$ $(m > 1)$ is much smaller than the set of $3^m \times 3^m$ fully projected matrices. In this section, we will improve the estimation of the upper bound of $p_{\mathrm{IV,V}}$ by constructing the iterations of $3^m \times 3^m$ fully projected matrices.

Definition 2.1. For $m \geqslant 0$, $k \geqslant 1$,

$$T_{m,k} := \left\{ w \in \{0, 1\}^{3^m k} \colon |w| := \sum_i > 0 \right\}$$
$$\Psi_{m,k} := T_{m,k}^{3^m}$$
$$\Phi_m(1) := \Psi_{m,1}$$

From the definitions, $\Psi_{m,k}$ denotes the set of $3^m \times 3^m k$ fully projected matrices, and $\Phi_1(1)$ is the same as the Φ in Lemma 1.10. Let $\Phi_m(0) = \{0, 1\}^{3^{2m}} \backslash \Phi_m(1)$. Iteration is the same as that in Section 1.4. We have, according to Lemma 1.7,

$$\Phi_m^n(0) = \{0, 1\}^{3^{2m}} \backslash \Phi_m^n(1), \qquad \forall m, n \geqslant 1$$

Proposition 2.2. For all $m, n \geqslant 1$, let

$$\pi_n^{(m)}(p) = \mathbb{P}[\sigma_p^{mn}(1) \in \Phi_m^n(1)]$$

Then

$$\pi_{n+1}^{(m)}(p) = \left[\sum_{i_1=1}^{3} C_3^{i_1} p^{i_1} (1-p)^{3-i_1} \left[\sum_{i_2=1}^{3i_1} C_{3i_1}^{i_2} p^{i_2} (1-p)^{3i_1 - i_2} \right. \right.$$
$$\times \left[\cdots \left[\sum_{i_{m-1}=1}^{3i_{m-2}} C_{3i_{m-2}}^{i_{m-1}} p^{i_{m-1}} (1-p)^{3i_{m-2} - i_{m-1}} \right. \right.$$
$$\left. \left. \left. \left. \times \left[\sum_{i_m=1}^{3i_{m-1}} C_{3i_{m-1}}^{i_m} (p\pi_n^{(m)}(p))^{i_m} (1 - p\pi_n^{(m)}(p))^{3i_{m-1} - i_m} \right]^3 \right]^3 \cdots \right]^3 \right]^3 \right]^3$$

where $i_0 = 1$.

Proof. For $m=1$, it follows from Proposition 1.8.

For $m>1$, it is not difficult to see that

$$\pi_1^{(m)}(p)=\left[\sum_{i_1=1}^{3} C_3^{i_1} p^{i_1}(1-p)^{3-i_1}\left[\sum_{i_2=1}^{3i_1} C_{3i_1}^{i_2} p^{i_2}(1-p)^{3i_1-i_2}\right.\right.$$
$$\times\left[\cdots\left[\sum_{i_{m-1}=1}^{3i_{m-2}} C_{3i_{m-2}}^{i_{m-1}} p^{i_{m-1}}(1-p)^{3i_{m-2}-i_{m-1}}\right.\right.$$
$$\left.\left.\left.\left.\times\left[\sum_{i_m=1}^{3i_{m-1}} C_{3i_{m-1}}^{i_m} p^{i_m}(1-p)^{3i_{m-1}-i_m}\right]^3\right]^3\cdots\right]^3\right]^3\right]^3$$

On the other hand,

$$\pi_{n+1}^{(m)}(p)=\sum_{v\in\Phi_{m-1}(1)}\mathbb{P}(\sigma_p^{m-1}(1)=v)\,\mathbb{P}(\sigma_p^{mn}(\sigma_p(v))\in\Phi_m^{n+1}(1))$$

To calculate $\mathbb{P}(\sigma_p^{mn}(\sigma_p(v))\in\Phi_m^{n+1}(1))$, we introduce some notations: for words $v, w\in\Phi_k(1)$, $k\geqslant 1$, $|w|=\sum_i w_i$, $b\leqslant w$ means $v_i\leqslant w_i$ for all i; $\tilde{v}\in\Phi_{k+1}(1)$ is obtained by replacing all 0's in v by (000000000), and all 1's by (111111111).

Note that $\sigma_p(v)\leqslant\tilde{v}$; for a fixed word $w\in\Phi_m(1)$, if $w\nleqslant\tilde{v}$, then $\sigma_p^{mn}(\sigma_p(v))\notin\Phi_m^n(w)$ for all n. But $\sigma_p^{mn}(\sigma_p(v))$ can be written as

$$\sigma_p^{mn}(\sigma_p(v))=\sigma_p^{mn}(u_1)\cdots\sigma_p^{mn}(u_N),\qquad N=3^{2m}$$

and $u_1,\dots,u_N$ are independent. Using the same method as in the proof of Proposition 1.8, we have

$$\mathbb{P}(\sigma_p^{mn}(\sigma_p(v))\in\Phi_m^{n+1}(1))=\sum_{\substack{w\in\Phi_m(1)\\ w\leqslant\tilde{v}}}(p\pi_n^{(m)}(p))^{|w|}\,(1-p\pi_n^{(m)}(p))^{9|v|-|w|}$$

In the similar way, we have

$$\pi_1^{(m)}(p)=\sum_{v\in\Phi_{m-1}(1)}\mathbb{P}(\sigma_p^{m-1}(1)=v)\sum_{\substack{w\in\Phi_m(1)\\ w\leqslant\tilde{v}}}p^{|w|}(1-p)^{9|v|-|w|}$$

Comparing the representation formulas of $\pi_1^{(m)}(p)$ and $\pi_{n+1}^{(m)}(p)$, we accomplish the proof.

It follows from Proposition 2.2 that for fixed m, if $n\uparrow$ or $p\downarrow$, then $\pi_n^{(m)}(p)\downarrow$. Hence the limit $\lim_{n\to\infty}\pi_n^{(m)}(p)$ exists. The problem is whether or not this limit is positive. Let

$$p_m=\inf\{p\colon \pi_\infty^{(m)}(p)=\lim_{n\to\infty}\pi_n^{(m)}(p)>0\}$$

Then we have the following result.

Lemma 2.3. For all $m \geqslant 1$, $p_{\mathrm{IV,V}} \leqslant p_m$.

Proof. Let A be a Mandelbrot percolation process with $A = \bigcap A_n$. Then

$$p_{\mathrm{IV,V}} = \inf\{p\colon \mathbb{P}[\lambda(\pi A) = 1] > 0\}$$

and

$$\lambda(\pi A) = 1 \Leftrightarrow \lambda(\pi A_n) = 1, \quad \forall n \geqslant 1 \Leftrightarrow \sigma_p^n(1) \in \Phi_n(1), \quad \forall n \geqslant 1$$

Hence

$$\mathbb{P}[\lambda(\pi A) = 1] = \lim_{n \to \infty} \pi_1^n(p)$$

For all $m \geqslant 1$, if $p > p_m$, then

$$0 < \lim_{n \to \infty} \pi_n^{(m)}(p) = \lim_{n \to \infty} \mathbb{P}[\sigma_p^{mn}(1) \in \Phi_m^n(1)]$$

$$\leqslant \lim_{n \to \infty} \mathbb{P}[\sigma_p^{mn}(1) \in \Phi_{mn}(1)] = \lim_{n \to \infty} \pi_1^{(mn)}(p)$$

$$= \mathbb{P}[\lambda(\pi A) = 1]$$

Hence $p > p_{\mathrm{IV,V}}$. ∎

Although we can define an iterate function $G_{p,m}(x)$ to find p_m as in Lemma 1.9, we estimate p_m by testing different values because of the lengthy expression of $(\partial/\partial x)\, G_{p,m}(x)$: for fixed p_0, if $\pi_\infty^{(m)}(p_0) > 0$, then $p_m \leqslant p_0$; if $\pi_\infty^{(m)}(p_0) = 0$, then $p_m > p_0$. Now have the estimations of p_m for $m \leqslant 7$ as in Table I.

Lemma 2.4. $p_{\mathrm{IV,V}} \leqslant 0.6317$.

Table I

m	p	$\pi_\infty^{(m)}(p)$
1	0.7307	0.797379
2	0.7056	0.716061
3	0.6869	0.626459
4	0.6712	0.530710
5	0.6569	0.432689
6	0.6437	0.348253
7	0.6317	0.270622

3. IMPROVING THE ESTIMATION OF THE LOWER BOUND OF $p_{\text{V,VI}}$

In this section, we will estimate the lower bound of $p_{\text{V,VI}}$ by using the methods of ordinary branching processes. Because the construction of the process is somewhat complex, let us start from a simple estimation. Lemma 3.1 was proved in Chayes *et al.*,[3] Section (b); we prove it again in order to introduce some useful notations.

In the following, a *level-n* square I^n_{kl} is said to be *open* if $\sigma^n_p(1)(k, l) = 1$; otherwise it is *closed.*

Lemma 3.1. $p_{\text{V,VI}} > 1/\sqrt{3}$.

Proof. Consider the segment $S = \{1/3\} \times [0, 1]$ in Fig. 1.

We say a *level-n* segment (common side which is in S of two level-n squares) passable if the two consecutive *level-n* squares are both open.

Let Z_n be the number of passable *level-n* segments in S. Then

$$EZ_1 = 3p^2$$

Hence for $p \leqslant 1/\sqrt{3}$, the branching process $\{Z_n\}_{n \geqslant 1}$ dies out with probablity 1. ∎

Lemma 3.2. $p_{\text{V,VI}} > 0.5917$.

Proof. Continue to use the notations in Lemma 3.1. Let S_1 denote the common side of I^1_{11} and I^1_{14}. If I^1_{11} and I^1_{14} are both open, we construct a branching process as follows.

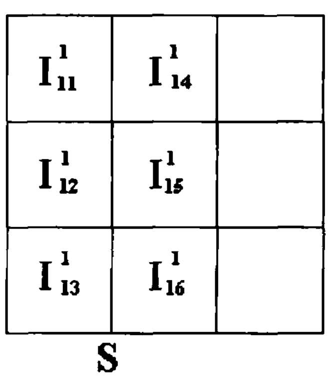

Fig. 1

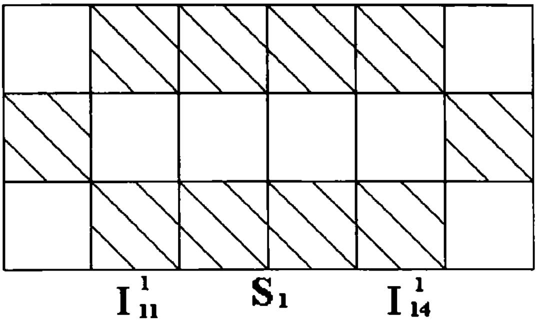

Fig. 2

In Fig. 2, we say a *level-n* segment ($n \geqslant 2$) in S_1 passable if there is a connected component in the two consecutive *level-*$(n-1)$ squares I_k^{n-1} and I_l^{n-1} generated by *level-n* open squares that crosses it and connects into left and right shadows. We can see in the figure that a *level-n* segment's passability is the same as in Lemma 3.1 when it is not in the middle of a *level-*$(n-1)$ segment.

Let Z_n denote the number of *level-*$(n+1)$ passable segments in S_1. Then

$$EZ_1 = 2p^2 + [p(1-(1-p)^2) + p(1-p)^2\, p(1-(1-p)^3)]^2$$

If $p < 0.591721$, then $EZ_1 < 1$. And for the common side S_2 of I_{12}^1 and I_{15}^1, S_3 of I_{13}^1 and I_{16}^1, we construct branching processes by the same method. Then the three processes are i.i.d. Hence they will all die out when $p \leqslant 0.5917$, i.e., the segment S cannot be crossed with probability 1. ∎

Lemma 3.3. $p_{\mathrm{V,VI}} > 0.6346$.

Proof. We divide S into three branching processes as in Lemma 3.2. For S_1, see Fig. 3. We say a *level-*$(2n+1)$ segment in S_1 is passable if there is a connected component in the two consecutive *level-*$(2n-1)$ squares I_k^{2n-1} and I_l^{2n-1} generated by *level-*$(2n+1)$ open squares that crosses it and connects into left and right shadows. Let Z_n denote the number of *level-*$(2n+1)$ passable segments in S_1. A computer calculation shows that if

$$p = 0.6346$$

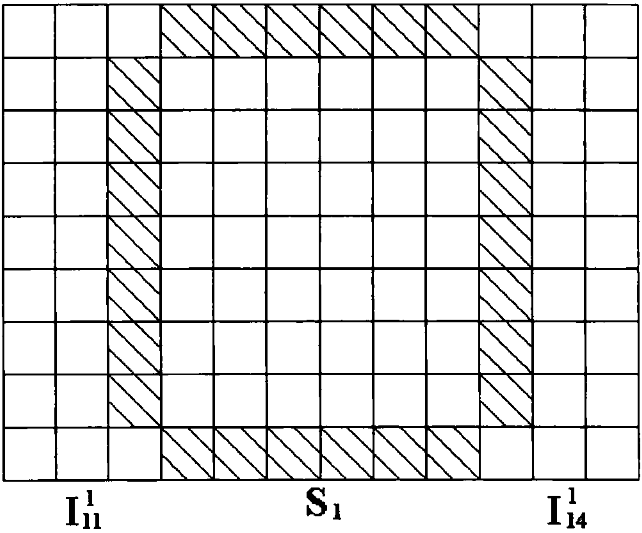

Fig. 3

then

$$EZ_1 = 0.999658$$

Hence $p_{V,VI} > 0.6346$. ∎

Proposition 3.4. The phase V in Mandelbrot percolation exists.

Proof. From Lemmas 2.4 and 3.3 we have

$$p_{IV,V} < p_{V,VI} \quad ∎$$

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